MATH 120A Prep: Complex Numbers II - Roots of Unity

Facts to Know:

Powers of Complex Numbers: $f(x) = x^n$

$$f(re^{i\theta}) = r^n e^{in\theta}$$

Unit Circle: $r = 1, \quad f(e^{i\theta}) = e^{in\theta}$ multiply angle by n.

* Roots are like dividing by n...

Roots of Unity: $\{$complex numbers $z : z^n = 1\}$ = nth roots of unity.

solving $z^n - 1 = 0$

Examples:

1. Find the nth roots of unity.

$$\{z : z^n = 1\}$$

$$z = re^{i\theta} \quad z^n = r^n e^{i\theta n} \quad 1 = 1 \cdot e^{i0}$$

$$r = 1$$

$$z = e^{i\theta} \quad \theta n = 0 \to \theta = 0 \to e^{i0} = 1$$

$$1 = e^{i0} = e^{i2\pi} = e^{i4\pi} = \dots = e^{-i2\pi} = \dots$$

$$1 = e^{i2k\pi} \quad k \in \mathbb{Z} \text{ integer}$$

$$z^n = e^{i\theta n} = 1 = e^{i2k\pi}$$

$$\theta n = 2\pi k \quad k \in \mathbb{Z} \qquad \theta = \frac{2\pi k}{n} \text{ for some } k \in \mathbb{Z}$$

nth roots of unity are $\left\{ e^{i\frac{2\pi k}{n}} : k \in \mathbb{Z} \right\}$

$2\pi = \frac{2\pi n}{n}$ only consider $k \in \{0, 1, \dots, n-1\}$

$$\boxed{n\text{th roots of unity} = \left\{ e^{i\frac{2\pi k}{n}} : k \in \{0, 1, \dots, n-1\} \right\}}$$

powers of $e^{i\frac{2\pi}{n}}$. $\left(e^{\frac{i2\pi}{n}}\right)^k = e^{\frac{2\pi k i}{n}}$

creates the rest of them by rotations of $\frac{2\pi}{n}$.

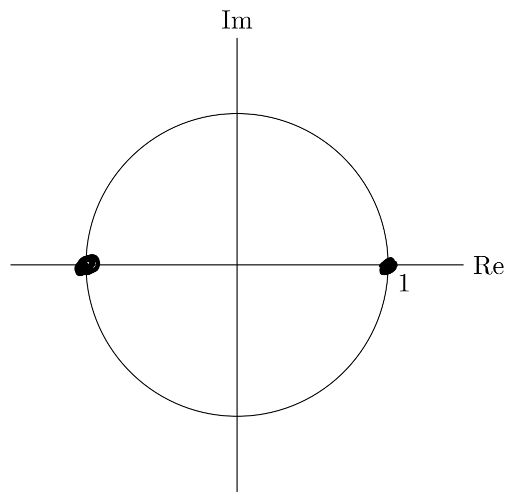
2. Sketch the 2nd, 3rd, 4th, and 6th roots of unity.

$(n=2)$

2nd roots of unity

$\left\{e^{i0}, e^{\frac{2\pi i}{2}} = e^{\pi i}\right\}$

$1 \qquad -1$

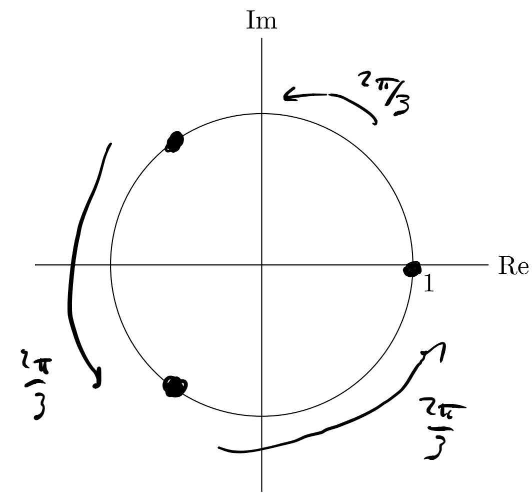


$(n=3)$

3rd roots of unity

$\left\{e^{i0}, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\right\}$

1

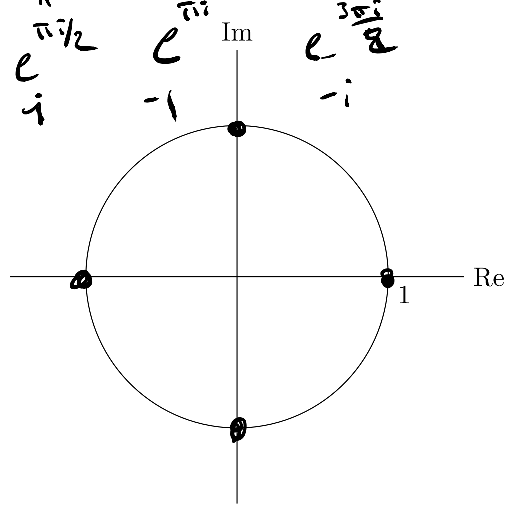


$(n=4)$

4th roots of unity

$\left\{e^{i0}, e^{\frac{2\pi i}{4}}, e^{\frac{4\pi i}{4}}, e^{\frac{6\pi i}{4}}\right\}$

$1 = 1$, $e^{\pi i/2} = i$, $e^{\pi i} = -1$, $e^{\frac{3\pi i}{2}} = -i$



$(n=6)$

6th roots of unity

$\left\{e^{i0}, e^{\frac{2\pi i}{6}}, e^{\frac{4\pi i}{6}}, e^{\frac{6\pi i}{6}}, e^{\frac{8\pi i}{6}}, e^{\frac{10\pi i}{6}}\right\}$

1 $\qquad e^{\pi i} = -1$